GRAMSCI KEYNES PRATO INSTITUTE

SUBJECT: MATHEMATICS
PROF. ANNALISA MALOSSI

Ed: this teaching unit was implemented and experimented during a course held in 2018/19 by Oxfam in Tuscany secondary schools (lower and upper).Some examples are in Italian.

Remarkable (notable) products

STRATIFIED LEARNING UNIT - REMARKABLE PRODUCTS

STEP 1 motivation	
	.Computation of remarkable products using only the normal rules of algebraic calculus (which the class already knows) • Calculation of remarkable products using only the normal rules of algebraic calculation (which the class already knows) • $(a+b)^2 = (a+b) \bullet (a+b) = \ldots$ • $(a+b)^3 = (a+b) \bullet (a+b) \bullet (a+b) = \ldots$ • $(a+b+c)^2 = (a+b+c) \bullet (a+b+c) = \ldots$ • $(a+b) \bullet (a-b) = \ldots$ • Activities (1) on the board and in small groups
STEP 2 INPUT PRESENTATION	
	For each of the remarkable products the teacher presents: • Notable product construction rules with graphic part • Examples Activity (2) to recognize the terms of the binomial/trinomial and calculate the terms of the remarkable product (in small groups)
STEP 3 FOCUS	
	For each of the remarkable products: (in small groups) • Multiple choice activity (3), activity (4) type V/F • Matching activities (5) with arrows
STEP 4 PRACTICE	
	For each of the remarkable products: • find and correct - activities (6) (in small groups) • Activity (7) of enunciation of the rules to construct the remarkable product (individual) (+)
STEP 5/6 FOLLOW UP	
	For all types of remarkable products: • Activities (8) construction of remarkable products (always in small groups)
STEP 7 ASSESSMENT	

	Test on the construction of remarkable products (individual)
	Once the teaching unit is finished, the next step will be the use of remarkable products in algebraic expressions

STRATIFIED LEARNING UNIT

remarkable products

STEP 1 MOTIVATION

Some types of multiplication, which happen to be encountered frequently, can be carried out quickly, remembering some simple rules, called REMARKABLE PRODUCTS. We're going to study four types. For each of these we proceed using the normal rules of algebraic calculation (which the class already knows).

1) The product of the sum of two monomials by their difference		
	$(a+b)\bullet(a-b)=$ $a^2-ab+ba-b^2=$ a^2-b^2	multiplying reducing similar terms
2) SQUARE OF A BINOMIAL:		
	$(a+b)^2=$ $(a+b)\bullet(a+b)=$ $a^2+ab+ba+b^2=$ $a^2+2ab+b^2$	PROPERTIES OF EXPONENTIATIONS multiplying reducing similar terms
3) SQUARE OF A TRINOMIAL:		
	$(a+b+c)^2=$ $(a+b+c)\bullet(a+b+c)=$ $a^2+ab+ac+ba+b^2+bc+ca+cb+c^2=$ $a^2+b^2+c^2+2ab+2ac+2bc$	PROPERTIES OF EXPONENTIATIONS multiplying reducing similar terms
4) CUBE OF A BINOMIAL:		
	$(a+b)^3=$ $(a+b)\bullet(a+b)^2=$ $(a+b)\bullet\left(a^2+2ab+b^2\right)=$ $a^3+2a^2b+ab^2+ba^2+2ab^2+b^3=$ $a^3+3a^2b+3ab^2+b^3=$	PROPERTIES OF EXPONENTIATIONS multiplying reducing similar terms

For each of the four products, examples are made on the board and then exercises in small groups, using only the normal rules of algebraic calculation.

Activity 1: Run the following products using the algebraic calculation rules you already know.

1) The product of the sum of two monomials by their difference		
	$(x + 3) \bullet (x - 3) =$ …	multiplying reducing similar terms
2) SQUARE OF A BINOMIAL:		
	$(3a - 2b)^2 =$ $(3a - 2b) \bullet (3a - 2b) =$ …	PROPERTIES OF EXPONENTIATIONS multiplying reducing similar terms
3) SQUARE OF A TRINOMIAL:		
	$(1 - 2x + y)^2 =$ $(1 - 2x + y) \bullet (1 - 2x + y) =$ …	PROPERTIES OF EXPONENTIATIONS multiplying reducing similar terms
4) CUBE OF A BINOMIAL:		
	$(x + 2y)^3 =$ $(x + 2y) \bullet (x + 2y) \bullet (x + 2y) =$ …	PROPERTIES OF EXPONENTIATIONS multiplying reducing similar terms

This long and tedious work is used to appreciate the savings of time and effort that can be achieved by knowing the rules of remarkable products.

STEP 2 INPUT PRESENTATION

(Repeats for each of the four notable products)

1) The product of the sum of two monomials by their difference

$$(A + B) \bullet (A - B) = A^2 - B^2$$

RULE

If A and B are two monomials, the product of the sum of A and B for their difference is equal to the difference between the square of A and the square of B.

EXAMPLE: (is the first of a series of examples to do on the board)

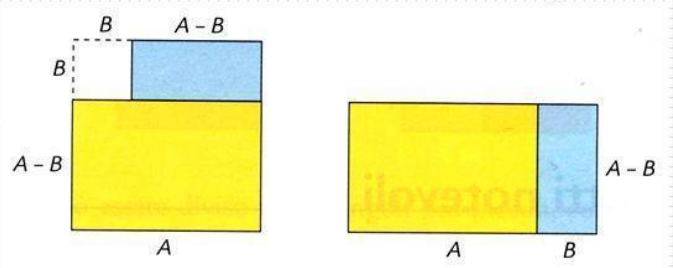
We calculate $\left(2x + y^2\right) \bullet \left(2x - y^2\right)$

$(A + B) \bullet (A - B) = A^2 - B^2$
$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$
$\left(2x + y^2\right) \bullet \left(2x - y^2\right) = (2x)^2 - \left(y^2\right)^2 = 4x^4 - y^4$

LET 'S VISUALIZE THE CONCEPT:

Equality $(A + B) \bullet (A - B) = A^2 - B^2$ can also be deduced graphically

Somma per differenza (interpretazione geometrica)



- Ritagliamo da un angolo di un quadrato di lato A un quadrato di lato B (a sinistra): si ottiene una figura di area A^2-B^2 che è l'unione del rettangolo giallo e del rettangolo azzurro.
- Ritagliamo ora il rettangolo azzurro e incolliamolo a quello giallo (a destra): si ottiene un nuovo rettangolo di lati A+B e A-B avente area (A+B)(A-B).
- Dall'equivalenza delle due figure segue $(A+B)(A-B)=A^2-B^2$.

2) SQUARE OF A BINOMIAL:

$$(A + B)^2 = A^2 + B^2 + 2AB$$

RULE

If A and B are two monomials, the square of $(A + B)$ is equal to the square of A, plus the square of B, plus the double product of A and B.

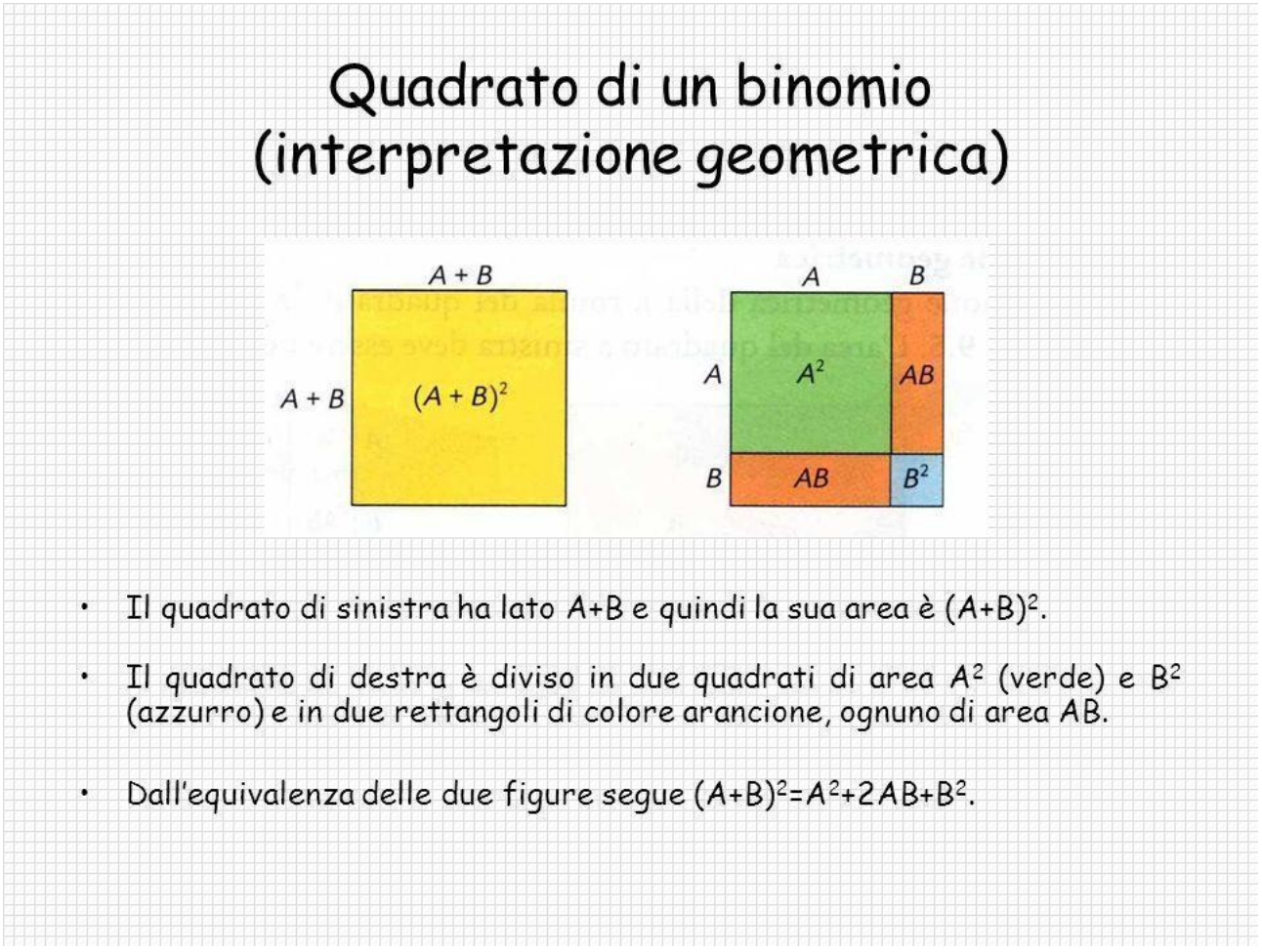
EXAMPLE: (is the first of a series of examples to do on the board)

We calculate $\left(2x + y^3\right)^2$

$(A + B)^2 = A^2 + B^2 + 2AB$
↓ ↓ ↓ ↓ ↓
$\left(2x + y^3\right)^2 = (2x)^2 + \left(y^3\right)^2 + 2(2x)\left(y^3\right) = 4x^2 + y^6 + 4xy^3$

LET 'S VISUALIZE THE CONCEPT:

Equality $(A + B)^2 = A^2 + B^2 + 2AB$ can also be deduced graphically



3) SQUARE OF A TRINOMIAL:

$$(A + B + C)^2 = A^2 + B^2 + C^2 + 2AB + 2AC + 2BC$$

RULE

If A, B, and C are three monomials, the square of $(A + B + C)$ is equal to the square of A, plus the square of B, plus the square of C, plus the double products of each of the three terms for all that follow it.

EXAMPLE: (is the first of a series of examples to do on the board)

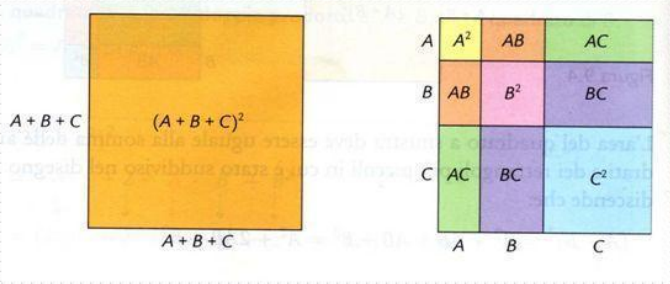
We calculate $\left(2x + y^3 - 2\right)^2$

$(A + B + C)^2 = A^2 + B^2 + C^2 + 2AB + 2AC + 2BC$
↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓
$\left(2x + y^3 - 2\right)^2 = (2x)^2 + \left(y^3\right)^2 + (-2)^2 + 2(2x)\left(y^3\right) + 2(2x)(-2) + 2\left(y^3\right)(-2) = 4x^2 +$

LET 'S VISUALIZE THE CONCEPT:

Equality $(A + B + C)^2 = A^2 + B^2 + C^2 + 2AB + 2AC + 2BC$ can also be deduced graphically

Quadrato di un trinomio (interpretazione geometrica)



- Il quadrato di sinistra ha lato A+B+C e quindi la sua area è $(A+B+C)^2$.
- Il quadrato di destra è diviso in un quadrato giallo di area A^2, uno rosa di area B^2 e uno azzurro di area C^2, in due rettangoli arancione ognuno di area AB, in due rettangoli verdi ognuno di area AC e in due rettangoli viola ognuno di area BC.
- Dall'equivalenza delle due figure segue $(A+B+C)^2=A^2+B^2+C^2+2AB+2AC+2BC$.

4) CUBE OF A BINOMIAL:

$$(A + B)^3 = A^3 + B^3 + 3A^2B + 3AB^2$$

RULE

If A and B are two monomials, the cube of $(A + B)$ is equal to the cube of A, plus the cube of B, plus the triple product of the square of A times B, plus the triple product of A times the square of B.

EXAMPLE: (is the first of a series of examples to do on the board)

We calculate $\left(2x + y^2\right)^3$

$(A + B)^3 = A^3 + B^3 + 3A^2B + 3AB^2$
↓ ↓ ↓ ↓ ↓ ↓
$\left(2x + y^2\right)^3 = (2x)^3 + \left(y^2\right)^3 + 3(2x)^2\left(y^2\right) + 3(2x) = 8x^3 + y^6 + 12x^2y^2 + 6xy^4$

FOR EACH OF THE REMARKABLE PRODUCTS: small group activities

Activity 2: Recognize the terms of the binomial/trinomial and calculate the terms of the remarkable product.

Activity 2.1 Follow the instructions to calculate the product of the sum of two monomials by their difference, completing the table by rows.

Sum by difference between two monomials	Calculates the square of the monomial that has the same sign in the two binomials	Calculate the square of the monomial that has a discordant sign in the two binomials	Write the difference between the first and second squares you calculated
$(A + B) \bullet (A - B)$	A^2	B^2	$A^2 - B^2$
$(3a + 5b) \bullet (3a - 5b)$	$(3a)^2 = 9a^2$	$(5b)^2 = 25b^2$	$9a^2 - 25b^2$
$(2x - 3y) \bullet (-2x - 3y)$	$(-3y)^2 = 9y^2$	$(2x)^2 = 4x^2$	$9y^2 - 4x^2$
⋮			

Task 2.2 Follow the instructions to calculate the square of a binomial, completing the table by rows.

SQUARE OF A BINOMIAL:	Locate the first monomial	Locate the second monomial	Calculate the square of the first monomial	Calculate the square of the second monomial	Calculate the double product	Write the algebraic sum of what you have calculated
$(A+B)^2$	A	B	A^2	B^2	$2AB$	A^2+B^2+2AB
$(-5a+b)^2$	$-5a$	b	$(-5a)^2=25a^2$	b^2	$-10ab$	$25a^2+b^2-10ab$
$\left(3x+\frac{1}{2}y\right)^2$						
$\vdots$						

Task 2.2 Follow the instructions to calculate the square of a binomial, completing the table by rows.

SQUARE OF A TRINOMIAL:	Calculate the square of the first monomial	Calculate the square of the second monomial	Calculates the square of the third monomial	Calculate double products	Write the algebraic sum of what you have calculated
$(A+B+C)^2$	A^2	B^2	C^2	$2AB+2AC+2BC$	$A^2+B^2+C^2 2AB+2AC+2BC$
$(-5a+b-3)^2$	$25a^2$	b^2	9	$-10ab+30a-6b$	$25a^2+b^2+9-10ab+30a-6b$
$\left(3x+\frac{1}{2}y-a\right)^2$					
$\vdots$					

Task 2.2 Follow the instructions to calculate the square of a binomial, completing the table by rows.

CUBE OF A BINOMIAL:	Calculates the cube of the first monomial	Calculates the cube of the second monomial	Calculates the triple product of the square of the first monomial for the second monomial	Calculate the triple product of the first monomial for the square of the second monomial	Write the algebraic sum of what you have calculated
$(A+B)^3$	A^3	B^3	$3A^2B$	$3AB^2$	$A^3+B^3+3A^2B+3AB^2$
$\left(3x+\frac{1}{2}y\right)^3$	$27a^3$	$\frac{1}{8}y^3$	$3(3x)^2\left(\frac{1}{2}y\right)=\frac{27}{2}x^2y$	$3(3x)\left(\frac{1}{2}y\right)^2=\frac{9}{4}xy^2$	$27a^3+\frac{1}{8}y^3+\frac{27}{2}x^2y+\frac{9}{4}xy^2$

STEP 3 FOCUSING

In this phase, exercises are carried out, always in small groups, on each single remarkable product (combination with arrows), but also on different remarkable products (multiple answer or true/false).

Activity 3: (combined with arrows) associate with arrows to each polynomial in the first column the remarkable product of which is the development

Activity 3.1 SUM BY DIFFERENCE (for the first binomial the exercise is carried out as an example)

a)	$4x^2 - y^2$	**A)**	$(2x - y)(2x + y)$
b)	$y^2 - 4x^2$	**B)**	$(-2x - y)(-2x + y)$
c)	$x^2 - 4y^2$	**C)**	$(-y - 2x)(-y + 2x)$
d)	$4y^2 - x^2$	**D)**	$(2y - x)(2y + x)$
		E)	$(x + 2y)(x - 2y)$
		F)	$(-x - 2y)(-x + 2y)$
		G)	$(y - 2x)(y + 2x)$

Activity 3.2 BINOMIAL SQUARE (for the first binomial the exercise is carried out as an example)

a)	$x^2 - 2xy + y^2$	**A)**	$(x - y)^2$
b)	$x^2 + 2xy + y^2$	**B)**	$(1 - 2a)^2$
c)	$1 - 4a + 4a^2$	**C)**	$(-1 - 2a)^2$
d)	$-1 - 4a - 4a^2$	**D)**	$-(1 + 2a)^2$
e)	$1 + 4a + 4a^2$	**E)**	$(2a - 1)^2$
		F)	$(y - x)^2$
		G)	$(-x - y)^2$

Activity 3.2 BINOMIAL SQUARE (for the first binomial the exercise is carried out as an example)

a)	$x^3 + 9x^2 + 27x + 27$	**A)**	$(3 - x)^3$
b)	$- x^3 - 9x^2 - 27x - 27$	**B)**	$(3 + x)^3$
c)	$x^3 - 9x^2 + 27x - 27$	**C)**	$(- 3 - x)^3$
d)	$- x^3 + 9x^2 - 27x + 27$	**D)**	$- (3 + x)^3$
		E)	$(x - 3)^3$

Activity 4: (multiple choice answers)

1. They are recognized as remarkable products	
A	Only the square and the binomial cube
B	All products between polynomials
C	Only the powers of binomial
D	Some products among polynomials with particular characteristics

2. The square polynomial of a binomial is	
A	Always a binomial
B	Always a quadrinomial
C	Always a trinomial
D	Always a second-degree trinomial

3. The degree of the square of a binomial is	
A	Always two
B	Always even
C	You can't say before performing the calculation
D	May be odd

4. In the square of a binomial the double product is	
A	Positive if the terms of the binomial agree
B	Always be positive!
C	Positive only if the terms of the binomial are positive
D	Positive if the terms of the binomial are discordant

5. By developing $\left(a-\frac{1}{2}\right)^2$ you get	
A	$a^2+\frac{1}{4}$
B	$a^2+\frac{1}{4}-1$
C	$a^2+\frac{1}{4}-2a$
D	$a^2+\frac{1}{4}-a$

6. The polynomial $x^2+\frac{1}{4}y^2+xy$ is the development of	
A	$\left(-x-\frac{y}{4}\right)^2$
B	$\left(-x+\frac{y}{2}\right)^2$
C	$\left(x-\frac{y}{2}\right)^2$
D	$\left(x+\frac{y}{2}\right)^2$

7. The product $(x-5)(-5-x)$ is	
A	x^2-25
B	$25-x^2$
C	$-x^2-25$
D	$(x-5)^2$

8. CUBE OF A BINOMIAL:	
A	A binomial with the coefficients multiplied by three
B	A quadrinomial
C	A trinomial
D	A quadrinomial with the coefficients multiplied by three

9. What is the development of $(a+2)^3$?	
A	a^3+8
B	a^3+8+6a
C	a^3+8-6a^2-12a
D	a^3+8+6a^2+12a

10. What is the development of $(a+b-1)^2$?	
A	a^2+b^2+1
B	$a^2+b^2+1-2a+2ab-2b$
C	$a^2+b^2+1-2a-2ab+2b$
D	a^2+b^2-1

Activity 5: (true/false)

A.	The square of the binomial $(a + 3b)$ is equal to $a^2 + 9b^2$	**V**	**F**
B.	Two opposing binomials have the same square	**V**	**F**
C.	The square of a polynomial is given by the product of the polynomial for itself	**V**	**F**
D.	Two opposite binomials have opposite cubes	**V**	**F**

A.	The product is $(a - x)(a + 2x)$ not a remarkable product	**V**	**F**
B.	The binomial cube $(a + 2b)$ is equal to $a^3 + 8b^3$	**V**	**F**
C.	The square of a trinomial is a trinomial	**V**	**F**
D.	The squares of two opposing binomas are opposite	**V**	**F**
E.	The cube of a binomial is a trinomial		

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STEP 4 PRACTICE

For each of the remarkable products, mistake finding and correction activities can be carried out (always in small groups)

Activity 6: Mistakes have been made in the calculation of the following significant products; individual and correctable.

Assets 6.1 SUM FOR DIFFERENCE

Wrong form	Error type	Exact Shape
$(x^3 - 2)(x^3 + 2) = x^9 - 4$	Incorrect square of the first term	$(x^3 - 2)(x^3 + 2) = x^6 - 4$
$(-3 - x)(x - 3) = x^2 - 9$		
$(x^2 - 1)(x^2 + 1) = x^4 - 2$		
⋮		

Activity 6.2 BINOMIAL/TRINOMIAL SQUARE

Wrong form	Error type	Exact Shape
$(a + b)^2 = a^2 + b^2$	The double product is missing	$(a + b)^2 = a^2 + 2ab + b^2$
$(a - b)^2 = a^2 - 2ab - b^2$		
$(-1 - x^2)^2 = -1 + 2x^2 - x^4$		
⋮		
$(a + b + 1)^2 = a^2 + b^2 + 1 + ab + a +$		
$(2a - b - 2)^2 = 4a^2 + b^2 + 4$		
⋮		

Activity 6.3 BINOMIAL CUBE

Wrong form	Error type	Exact Shape
$(a+b)^3 = a^3 + b^3$	Triple products are missing	$(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$
$(a-b)^3 = a^3 + 3a^2b + 3ab^2 - b^3$		
$(x-y)^3 = x^3 - y^3$		
⋮		

Activity 7: In this phase you can also do (individual) activities of enunciation of the rules to build the remarkable product (only for students without particular linguistic needs).

STEP 5/6 LANGUAGE AND FOLLOW UP

Activity 8: Construction of significant products (two-tier application exercises)
The cards are on two levels: (always in small groups)

- Separate for each major product type (Tables 8.1)
- All remarkable products together (Table 8.2)

Task 8.1.1 SUM BY DIFFERENCE Complete the table . Read the results aloud to each other (pairs)

Sum by Difference	Calculate the sum by difference
$\left(\frac{1}{2}x^2 - 2a\right)\left(\frac{1}{2}x^2 + 2a\right)$	
$\left(-a - \frac{5}{2}\right)\left(\frac{5}{2} - a\right)$	
$\left(x^3 + y\right)\left(x^3 - y\right)$	
⋮	

Task 8.1.2 BINOMIAL SQUARE Complete the table.Read the results aloud to each other (pairs)

SQUARE OF A BINOMIAL:	**Calculate the square of the binomial**
$(3ab - 2x^3)^2$	
$\left(\frac{2}{3}x - \frac{3}{2}\right)^2$	
$\left(\frac{3}{2}x^2 - 2\right)^2$	
⋮	

Task 8.1.3 TRINOMIAL SQUARE Complete the table.Read the results aloud to each other (pairs)

SQUARE OF A TRINOMIAL:	**Calculate the square of the trinomial**
$(1 + 3ab - 2x^3)^2$	
$\left(\frac{2}{3}x - 1 - \frac{3}{2}\right)^2$	
$\left(\frac{3}{2}x^2 - 2 + x\right)^2$	
⋮	

Activity 8.1.4 BINOMIAL CUBE Complete the table.Read the results aloud to each other (pairs)

CUBE OF A BINOMIAL:	**Calculate the binomial cube**
$(3ab - 2x^3)^3$	

$\left(\frac{2}{3}x - \frac{3}{2}\right)^3$	
$\left(\frac{3}{2}x^2 - 1\right)^3$	
⋮	

Activities 8.2 REMARKABLE PRODUCTS Complete the table (some fields have been filled as example) - Read the results aloud to each other (pairs)

Expression	**Product type considerable**	**Calculation of the remarkable product**
$\left(\frac{2}{3}x - 1 - \frac{3}{2}\right)^2$	SQUARE OF A TRINOMIAL:	
$\left(\frac{3}{2}x^2 - 1\right)^3$		
$\left(-a - \frac{5}{2}\right)\left(\frac{5}{2} - a\right)$	Sum by Difference	
$\left(-a + \frac{5}{2}\right)\left(\frac{5}{2} - a\right)$	SQUARE OF A BINOMIAL: $\left(\frac{5}{2} - a\right)^2$	
$(a+b)(a+b)(a+b)$	CUBE OF A BINOMIAL: $(a+b)^3$	
⋮		

STEP 7 ASSESSMENT

Test on the construction of remarkable products (individual)

The test can also be done at different levels:

- Simpler, such as type 8.1 tables, grouping the various types of remarkable product and presenting them in standard form
- More complex, as the table of type 8.2, asking to recognize the type of product remarkable and presenting it also in non-standard form.

Test Example: Calculates the following remarkable products.

1) $\left(3x - \frac{1}{2}y\right)\left(3x + \frac{1}{2}y\right)$

2) $\left(-\frac{1}{2}x + \frac{3}{2}y\right)\left(\frac{1}{2}x + \frac{3}{2}y\right)$

3) $(2a - b)(-2a - b)$

4) $\left(\frac{1}{3}x + \frac{3}{2}y\right)^2$

5) $\left(\frac{1}{2}a^2 - 3b^3\right)^2$

6) $\left(-\frac{1}{3}a^2 - \frac{2}{3}b\right)^2$

7) $\left(3x - \frac{1}{2}y + 2\right)^2$

8) $\left(a^2 - 2a + 1\right)^2$

9) $\left(-x^3 - \frac{1}{2}x + 2\right)^2$

10) $\left(\frac{1}{3}x^2 + 2y\right)^3$

11) $\left(\frac{1}{2}a - 2b\right)^3$

12) $\left(-3a - \frac{1}{3}b\right)^3$

13) $\left(\frac{1}{2}a - 1\right)^4$

14) $\left(\frac{1}{2}x - 2\right)\left(\frac{1}{4}x^2 + 4\right)\left(\frac{1}{2}x + 2\right)$

15) $\left(\frac{1}{2}a + 3b\right)^2\left(\frac{1}{2}a - 3b\right)^2$

16) $(a - 2b)^3(a + 2b)^3$